

On heavy paths in 2-connected weighted graphs*

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Abstract

A weighted graph is a graph in which every edge is assigned a non-negative real number. In a weighted graph, the weight of a path is the sum of the weights of its edges, and the weighed degree of a vertex is the sum of the weights of the edges incident with it. In this paper we give three weighted degree conditions for the existence of heavy or Hamilton paths with one or two given end-vertices in 2-connected weighted graphs.

Keywords: Weighted graph; Heavy path; Weighed degree

1 Introduction

We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only.

A *weighted graph* is a graph in which every edge e is assigned a non-negative real number $w(e)$, called the *weight* of e . Let $G = (V, E)$ be a weighted graph. For a subgraph H of G , $V(H)$ and $E(H)$ denote the sets of vertices and edges of H , respectively. The *weight* of H is defined by

$$w(H) = \sum_{e \in E(H)} w(e).$$

For a vertex $v \in V$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of vertices in H that are adjacent to v . We define the *weighted degree* of v in H by

$$d_H^w(v) = \sum_{x \in N_H(v)} w(vx).$$

When no confusion occurs, we will denote $N_G(v)$, $d_G(v)$ and $d_G^w(v)$ by $N(v)$, $d(v)$ and $d^w(v)$, respectively.

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An unweighted graph can be regarded as a weighted graph in which each edge e is assigned weight $w(e) = 1$. Thus, in an unweighted graph, $d^w(v) = d(v)$ for every vertex v , and the weight of a subgraph is simply the number of its edges.

About twenty years ago, Bondy and Fan [1] began the study of the existence of heavy paths and cycles in weighted graphs. They showed two results with Dirac-type weighted degree condition. In the following by an x -path we mean a path whose initial vertex is x ; and by an (x, y) -path we mean one whose end-vertices are x and y .

Theorem 1 (Bondy and Fan [1]). *Let G be a 2-connected weighted graph, x and y be two vertices of G , and d be a real number. If $d^w(v) \geq d$ for every vertex v in $V(G) \setminus \{x, y\}$, then G contains an (x, y) -path of weight at least d .*

Theorem 2 (Bondy and Fan [1]). *Let G be a 2-connected weighted graph and d be a real number. If $d^w(v) \geq d$ for every vertex v in $V(G)$, then either G contains a cycle of weight at least $2d$ or every heaviest cycle in G is a Hamilton cycle.*

Corresponding Ore-type conditions for the existence of heavy paths and cycles in weighted graphs were also obtained.

Theorem 3 (Enomoto, Fujisawa and Ota [4]). *Let G be a 2-connected weighted graph, x and y be two vertices of G , and d be a real number. If $d^w(v_1) + d^w(v_2) \geq 2d$ for every pair of nonadjacent vertices v_1 and v_2 in $V(G) \setminus \{x, y\}$, then G contains either an (x, y) -path of weight at least d or a Hamilton (x, y) -path.*

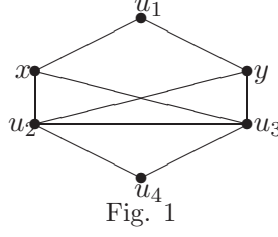
Theorem 4 (Bondy et al. [3]). *Let G be a 2-connected weighted graph and d be a real number. If $d^w(v_1) + d^w(v_2) \geq 2d$ for every pair of nonadjacent vertices v_1 and v_2 in $V(G)$, then G contains either a cycle of weight at least $2d$ or a Hamilton cycle.*

It can be seen that Theorems 2 and 4 generalize the classical results of Dirac and Ore on the existence of long cycles in unweighted graphs, respectively. Another well-known result on the existence of long cycles in unweighted graphs is the so-called Fan-type condition [5]. By constructing some examples, Zhang et al. [7] found that the Fan's condition cannot be generalized to weighted graphs directly. As one can expect, the corresponding condition for heavy paths in weighted graphs is also not valid.

False Statement 1. *Let G be a 2-connected weighted graph, x and y be two vertices of G , and d be a real number. If $\max\{d^w(v_1), d^w(v_2)\} \geq d$ for every pair of vertices v_1 and v_2 in $V(G) \setminus \{x, y\}$ with distance 2, then G contains either an (x, y) -path of weight at least d or a Hamilton (x, y) -path.*

This can be shown by the graph in Fig. 1. We assign weight 2 to the edge u_2u_3 , and weight 1 to all the remaining edges. Then, the resulting weighted graph is 2-connected

and $\max\{d^w(v_1), d^w(v_2)\} \geq 5$ for every pair of vertices v_1 and v_2 in $V(G) \setminus \{x, y\}$ with distance 2. Whereas, the graph contains neither an (x, y) -path of weight at least 5 nor a Hamilton (x, y) -path.



Here we strengthen the condition of False Statement 1 and get the following result:

Theorem 5. *Let G be a 2-connected weighted graph, x and y be two vertices of G , and d be a real number. If $\max\{d^w(v_1), d^w(v_2)\} \geq d$ for every pair of nonadjacent vertices v_1 and v_2 in $V(G) \setminus \{x, y\}$, then G contains either an (x, y) -path of weight at least d or a Hamilton (x, y) -path.*

In fact, a corresponding result on the existence of heavy cycles in weighted graphs was obtained by Fujisawa [6].

Theorem 6 (Fujisawa [6]). *Let G be a 2-connected weighted graph and d be a real number. If $\max\{d^w(v_1), d^w(v_2)\} \geq d$ for every pair of nonadjacent vertices v_1 and v_2 in $V(G)$, then G contains either a cycle of weight at least $2d$ or a Hamilton cycle.*

If we weaken the weighted degree sum condition of two nonadjacent vertices in Theorem 3 by that of three pairwise nonadjacent vertices, can we still have the same result? The answer is also negative.

False Statement 2. *Let G be a 2-connected weighted graph, x and y be two vertices of G , and d be a real number. If $d^w(v_1) + d^w(v_2) + d^w(v_3) \geq 3d$ for every three pairwise nonadjacent vertices v_1, v_2 and v_3 in $V(G) \setminus \{x, y\}$, then G contains either an (x, y) -path of weight at least d or a Hamilton (x, y) -path.*

This can be shown as follows. Let H_1 and H_2 be two complete graphs on at least three vertices with two vertices x and y in common. Let $G = H_1 \cup H_2$ and assign weight 0 to every edge of G . Then G is 2-connected and there exist no three pairwise nonadjacent vertices in G . So G satisfies the condition of False Statement 2 for any $d > 0$, but contains neither an (x, y) -path of weight at least d nor a Hamilton (x, y) -path.

When specifying only one end-vertex of the path, we have the following result:

Theorem 7. *Let G be a 2-connected weighted graph, x be a vertex of G , and d be a real number. If $d^w(v_1) + d^w(v_2) + d^w(v_3) \geq 3d$ for every three pairwise nonadjacent vertices v_1, v_2 and v_3 in $V(G) \setminus \{x\}$, then G contains either an x -path of weight at least d or a Hamilton x -path.*

Instead of Theorem 7, we will prove the following stronger result.

Theorem 8. *Let G be a 2-connected weighted graph, x be a vertex of G , and d be a real number. If $\max\{d^w(v_1), d^w(v_2), d^w(v_3)\} \geq d$ for every three pairwise nonadjacent vertices v_1, v_2 and v_3 in $V(G) \setminus \{x\}$, then G contains either an x -path of weight at least d or a Hamilton x -path.*

Further, under the condition of False Statement 2, we can prove the following result:

Theorem 9. *Let G be a 2-connected weighted graph, x and y be two vertices of G , and d a real number. If $d^w(v_1) + d^w(v_2) + d^w(v_3) \geq 3d$ for every three pairwise nonadjacent vertices v_1, v_2 and v_3 in $V(G) \setminus \{x, y\}$, then at least one of the following holds:*

- (1) G contains an (x, y) -path P with $w(P) \geq d$;
- (2) G contains an x -path P_1 and a y -path P_2 which are disjoint with $w(P_1) + w(P_2) \geq d$;
- (3) G contains an x -path P_1 and a y -path P_2 which are disjoint with $V(G) = V(P_1) \cup V(P_2)$.

Instead of Theorem 9, we will prove the following stronger result.

Theorem 10. *Let G be a 2-connected weighted graph, x and y be two vertices of G , and d be a real number. If $\max\{d^w(v_1), d^w(v_2), d^w(v_3)\} \geq d$ for every three pairwise nonadjacent vertices v_1, v_2 and v_3 in $V(G) \setminus \{x, y\}$, then at least one of the following holds:*

- (1) G contains an (x, y) -path P with $w(P) \geq d$;
- (2) G contains an x -path P_1 and a y -path P_2 which are disjoint with $w(P_1) + w(P_2) \geq d$;
- (3) G contains an x -path P_1 and a y -path P_2 which are disjoint with $V(G) = V(P_1) \cup V(P_2)$.

We give the proofs of Theorems 5, 8 and 10 in the following sections.

2 Proof of Theorem 5

If $d = 0$, then the assertion is obvious. Hence we may assume that $d > 0$. Let $|V(G)| = n$. We use induction on n .

If $n = 3$, then the result is trivially true. Suppose now that $n \geq 4$ and the theorem is true for all graphs on fewer than n vertices. Let $H = G - x$.

Case 1. H is 2-connected.

Since G is 2-connected, we have $d(x) \geq 2$. Choose a vertex $x' \in N(x) \setminus \{y\}$ such that $w(xx') = \max\{w(xv) : v \in N(x) \setminus \{y\}\}$. Then for every $v \in V(H) \setminus \{y\}$, we have $d_H^w(v) \geq d_G^w(v) - w(xx')$. Hence,

$$\max\{d_H^w(v_1), d_H^w(v_2)\} \geq d - w(xx')$$

for every pair of nonadjacent vertices v_1 and v_2 in $V(H) \setminus \{x', y\}$. By the induction hypothesis, there is an (x', y) -path P' in H such that either $w(P') \geq d - w(xx')$ or P' is a Hamilton path of H . Thus the path $P = xx'P'$ is a required path.

Case 2. H is separable.

Let z be a cut vertex of H , H_1 be a component of $H - z$, and $H_2 = H - z - H_1$. For $i = 1, 2$, let $G_i = G[H_i \cup \{x, z\}]$. If $xz \notin E(G_i)$, we add the edge xz of weight 0 to G_i . Now the two resulting graphs G_1 and G_2 are both 2-connected.

Case 2.1. $y = z$.

Let $d^w(v_0) = \min\{d^w(v) : v \in V(H_1) \cup V(H_2)\}$. Without loss of generality, we can assume that $v_0 \in V(H_1)$. Since for all $v \in V(H_2)$, $vv_0 \notin E(G)$, we have $d^w(v) = \max\{d^w(v), d^w(v_0)\} \geq d$. Then $d_{G_2}^w(v) = d^w(v) \geq d$ for all $v \in V(G_2) \setminus \{x, y\}$. By Theorem 1, there exists an (x, y) -path P in G_2 such that $w_{G_2}(P) \geq d$. It is obvious that P is not the added edge xy itself, so we can take the path P as a required path.

Case 2.2. $y \neq z$.

Without loss of generality, we can assume that $y \in V(H_1)$. It is easy to know that $\max\{d_{G_2}^w(v_1), d_{G_2}^w(v_2)\} \geq d$ for every pair of nonadjacent vertices in $V(G_2) \setminus \{x, z\}$. By the induction hypothesis, there exists an (x, z) -path P_2 in G_2 such that either $w_{G_2}(P_2) \geq d$ or P_2 is a Hamilton path in G_2 . It is obvious that P_2 is not the added edge xy itself.

Suppose that $w_{G_2}(P_2) \geq d$. Let P_1 be a (z, y) -path not passing through x in G_1 . Then the path $P = P_2P_1$ is a required path.

Suppose now that every (x, z) -path in G_2 has weight less than d . Then P_2 is a Hamilton path in G_2 , and it follows from Theorem 1 that there exists at least one vertex in H_2 with weighted degree less than d . Thus, every vertex in $V(H_1) \setminus \{y\}$ has weighted degree at least d .

Case 2.2.1. H_2 has at least two vertices.

Let G'_1 be the weighted graph such that $V(G'_1) = V(G_1) \cup \{x'\}$, where $x' \notin V(G)$; $E(G'_1) = E(G_1) \cup \{xx', x'z\}$; and

$$w_{G'_1}(e) = \begin{cases} w_{G_1}(e), & \text{if } e \in E(G_1); \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to know that G'_1 is 2-connected. Besides, we can see that each vertex in $V(G'_1) \setminus \{x, y\}$ other than x' and z has weighted degree at least d . Thus,

$$\max\{d_{G'_1}^w(v_1), d_{G'_1}^w(v_2)\} \geq d$$

for every pair of nonadjacent vertices v_1 and v_2 in $V(G'_1) \setminus \{x, y\}$. By the induction hypothesis, there exists an (x, y) -path P' in G'_1 such that either $w_{G'_1}(P') \geq d$ or P' is a Hamilton path in G'_1 .

Suppose that $w_{G'_1}(P') \geq d$. If P' contains neither the added edge xz nor the subpath $xx'z$, then it is a required path. Otherwise, we can use any (x, z) -path in G_2 to replace the added edge xz or the subpath $xx'z$, and then obtain an (x, y) -path in G with weight at least $w_{G'_1}(P') \geq d$.

Suppose that P' is a Hamilton path in G'_1 . Because $d_{G'_1}(x') = 2$, P' contains the subpath $xx'z$. Thus, we can use the path P_2 to replace the subpath $xx'z$, and then obtain a Hamilton (x, y) -path in G .

Case 2.2.2. H_2 has only one vertex.

Let x' be the vertex in H_2 . Then we have $x' \in N(x) \setminus \{y\}$, $d(x') = 2$ and $d^w(x') < d$. By a similar proof, we can obtain that there exists a vertex y' in $N(y) \setminus \{x\}$ such that $d(y') = 2$ and $d^w(y') < d$. Since for all vertex $v \in V(H_1) \setminus \{y\}$, $d^w(v) \geq d$, we have $y' = z$ and $d(z) = 2$.

If H_1 has only one vertex y , then $P = xx'zy$ is a Hamilton (x, y) -path in G . Thus, we can assume that H_1 has at least one vertex other than y . Let $G'_1 = G[H_1 \cup \{x\}]$. If $xy \notin E(G'_1)$, we add the edge xy of weight zero to G'_1 , and then get that G'_1 is 2-connected and $w_{G'_1}(v) = w_G(v) \geq d$ for every vertex $v \in V(G'_1) \setminus \{x, y\}$. By Theorem 1, there exists an (x, y) -path P in G'_1 such that $w_{G'_1}(P) \geq d$. It is obvious that P is not the added edge xy itself. So we can take the path P as a required path.

The proof is complete. \square

3 Proof of Theorem 8

If $d = 0$, then the assertion is obvious. Hence we may assume that $d > 0$. Let $|V(G)| = n$. We use induction on n .

If $n = 3$, then the result is trivially true. Suppose now that $n \geq 4$ and the theorem is true for all graphs on fewer than n vertices. Let $H = G - x$.

Case 1. H is 2-connected.

Since G is 2-connected, we have $d(x) \geq 2$. Choose $x' \in N(x)$ such that $w(xx') = \max\{w(xv) : v \in N(x)\}$. Then for every $v \in V(H)$, we have $d_H^w(v) \geq d_G^w(v) - w(xx')$. Hence,

$$\max\{d_H^w(v_1), d_H^w(v_2), d_H^w(v_3)\} \geq d - w(xx')$$

for every three nonadjacent vertices v_1, v_2 and v_3 in $V(H) \setminus \{x'\}$. By the induction hypothesis, there is an x' -path P' in H such that either $w(P') \geq d - w(xx')$ or P' is a Hamilton path of H . Then the path $P = xx'P'$ is a required path.

Case 2. H is separable.

Let y be a cut vertex of H , H_1 be a component of $H - y$, and $H_2 = H - y - H_1$. For $i = 1, 2$, let $G_i = G[H_i \cup \{x, y\}]$. If $xy \notin E(G_i)$, we add the edge xy of weight 0 to G_i . Now the two resulting graphs G_1 and G_2 are both 2-connected.

If for some $i \in \{1, 2\}$, $d_{G_i}^w(v) = d^w(v) \geq d$ for all v in $V(G_i) \setminus \{x, y\}$, then by Theorem 1, there is an (x, y) -path P such that $w(P) \geq d$. It is obvious that P is not the added edge xy itself, so we can take the path P as a required path.

Otherwise, for $i \in \{1, 2\}$, there exists a vertex v in $V(G_i) \setminus \{x, y\}$ such that $d_{G_i}^w(v) = d^w(v) < d$. Then for $i \in \{1, 2\}$, $\max\{d_{G_i}^w(v_1), d_{G_i}^w(v_2)\} \geq d$ for every pair of nonadjacent vertices v_1 and v_2 in $V(G_i) \setminus \{x, y\}$. By Theorem 5, there exists an (x, y) -path P_i such that either $w_{G_i}(P_i) \geq d$ or P_i is a Hamilton path of G_i . It is obvious that P_i is not the added edge xy itself.

If for some $i \in \{1, 2\}$, $w_{G_i}(P_i) \geq d$, then we can take the path P_i as a required path. Otherwise, P_i is a Hamilton (x, y) -path of G_i , and $C = P_1 P_2$ is a Hamilton cycle of G . Then G contains a Hamilton x -path.

The proof is complete. \square

4 Proof of Theorem 10

If $d = 0$, then the assertion is obvious. Hence we may assume that $d > 0$. Let $|V(G)| = n$. We use induction on n .

If $n = 3$, then the result is trivially true. Suppose now that $n \geq 4$ and the theorem is true for all graphs on fewer than n vertices. Let $H = G - x$.

Case 1. H is 2-connected.

Since G is 2-connected, we have $d(x) \geq 2$. Choose a vertex $x' \in N(x) \setminus \{y\}$ such that $w(xx') = \max\{w(xv) : v \in N(x) \setminus \{y\}\}$. Then for every $v \in V(H) \setminus \{y\}$, we have $d_H^w(v) \geq d_G^w(v) - w(xx')$. Hence,

$$\max\{d_H^w(v_1), d_H^w(v_2), d_H^w(v_3)\} \geq d - w(xx')$$

for every three pairwise nonadjacent vertices v_1, v_2 and v_3 in $V(H) \setminus \{x', y\}$. By the induction hypothesis, H satisfies the conclusion of the theorem. If H contains an (x', y) -path P' such that $w(P') \geq d - w(xx')$, then $P = xx'P'$ is an (x, y) -path in G with weight at least d . If G contains an x' -path P'_1 and a y -path P'_2 which are disjoint and have weight sum at least $d - w(xx')$, then the x -path $P_1 = xx'P'_1$ and the y -path $P_2 = P'_2$ are disjoint and have weight sum at least d . If G contains an x' -path P'_1 and a y -path P'_2 which are disjoint and contain all vertices of H , then the x -path $P_1 = xx'P'_1$ and the y -path $P_2 = P'_2$ are disjoint and contain all vertices of G .

Case 2. H is separable.

Let z be a cut vertex of H , H_1 be a component of $H - z$, and $H_2 = H - z - H_1$. For $i = 1, 2$, let $G_i = G[H_i \cup \{x, z\}]$. If $xz \notin E(G_i)$, we add the edge xz of weight 0 to G_i . Now the two resulting graphs G_1 and G_2 are both 2-connected.

Case 2.1. $y = z$.

If for some $i \in \{1, 2\}$, $d_{G_i}^w(v) = d^w(v) \geq d$ for all v in $V(G_i) \setminus \{x, y\}$, then by Theorem 1, there is an (x, y) -path P such that $w_{G_i}(P) \geq d$. It is obvious that P is not the added edge xy itself, so we can take the path P as a required path.

Otherwise, for $i \in \{1, 2\}$, there exists a vertex v in $V(G_i) \setminus \{x, y\}$ such that $d_{G_i}^w(v) = d^w(v) < d$. Then for $i \in \{1, 2\}$, we have $\max\{d_{G_i}^w(v_1), d_{G_i}^w(v_2)\} \geq d$ for every pair of nonadjacent vertices v_1 and v_2 in $V(G_i) \setminus \{x, y\}$. By Theorem 5, there exists an (x, y) -path P'_i such that either $w_{G_i}(P'_i) \geq d$ or P'_i is a Hamilton path of G_i . It is obvious that P'_i is not the added edge xy itself.

If for some $i \in \{1, 2\}$, $w_{G_i}(P'_i) \geq d$, we can take the path P'_i as a required path. Otherwise, P'_i is a Hamilton (x, y) -path of G_i , and $C = P'_1 P'_2$ is a Hamilton cycle of G . Thus G contains an x -path and a y -path which are disjoint and contain all vertices of G .

Case 2.2. $y \neq z$.

Without loss of generality, we can assume that $y \in V(H_1)$. It is easy to know that $\max\{d_{G_2}^w(v_1), d_{G_2}^w(v_2), d_{G_2}^w(v_3)\} \geq d$ for every three pairwise nonadjacent vertices in $V(G_2) \setminus \{x, z\}$. By the induction hypothesis, G_2 satisfies the conclusion of the theorem.

If G_2 contains an (x, z) -path P'_2 with weight at least d , then let P'_1 be a (z, y) -path not passing through x in G_1 . Then the path $P = P'_2 P'_1$ is an (x, y) -path in G with weight at least $w(P'_2) \geq d$. So we assume that every (x, z) -path in G_2 has weight less than d . Then, by Theorem 1, there exists at least one vertex in $V(H_2)$ with weighted degree less than d .

If G_2 contains an x -path P'_{21} and a z -path P'_{22} which are disjoint and have weight sum at least d , then let P'_1 be a (y, z) -path not passing through x in G_1 . So the x -path path $P_1 = P'_{21}$ and the y -path $P_2 = P'_1 P'_{22}$ are disjoint and have weight sum at least d .

So now we assume that G_2 contains an x -path P'_{21} and a z -path P'_{22} which are disjoint and contain all vertices of G_2 .

Case 2.2.1. $d_{G_1}^w(v) \geq d$ for all v in $V(H_1) \setminus \{y\}$.

Let G'_1 be the weighted graph such that $V(G'_1) = V(G_1) \cup \{x'\}$, where $x' \notin V(G)$; $E(G'_1) = E(G_1) \cup \{xx', x'z\}$; and

$$w_{G'_1}(e) = \begin{cases} w_{G_1}(e), & \text{if } e \in E(G_1); \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to know that G'_1 is 2-connected. Besides, we can see that each vertex in $V(G'_1) \setminus \{x, y\}$ other than x' and z has weighted degree at least d . Thus,

$$\max\{d_{G'_1}^w(v_1), d_{G'_1}^w(v_2)\} \geq d$$

for every pair of nonadjacent vertices v_1 and v_2 in $V(G'_1) \setminus \{x, y\}$. By Theorem 5, there exists an (x, y) -path P' in G'_1 such that either $w_{G'_1}(P') \geq d$ or P' is a Hamilton path in G'_1 .

Suppose that $w_{G'_1}(P') \geq d$. If P' contains neither the added edge xz nor the subpath $xx'z$, then it is an (x, y) -path in G with weight at least d . Otherwise, we can use any (x, z) -path in G_2 to replace the added edge xz or the subpath $xx'z$, and then obtain an (x, y) -path in G with weight at least $w_{G'_1}(P') \geq d$.

Suppose that P' is a Hamilton path in G'_1 . Because $d_{G'_1}(x') = 2$, P' contains the subpath $xx'z$. Let $P'_1 = P' - \{xx', x'z\}$. So the x -path $P_1 = P'_{21}$ and the y -path $P_2 = P'_1 P'_{22}$ are disjoint and contain all vertices of G .

Case 2.2.2. There exists a vertex in $V(H_1) \setminus \{y\}$ with weighted degree less than d .

Now, we have that $\max\{d^w(v_1), d^w(v_2)\} \geq d$ for every pair nonadjacent vertices v_1 and v_2 in $V(H_1) \setminus \{y\}$, and $\max\{d^w(v_1), d^w(v_2)\} \geq d$ for every pair nonadjacent vertices v_1 and v_2 in $V(H_2)$.

Recall that we have assumed that every (x, z) -path in G_2 has weight less than d . By Theorem 5, there exists a Hamilton (x, z) -path P'_2 in G_2 .

Case 2.2.2.1. H_2 has at least two vertices.

Let G'_1 be the weighted graph such that $V(G'_1) = V(G_1) \cup \{x'\}$, where $x' \notin V(G)$; $E(G'_1) = E(G_1) \cup \{xx', x'z\}$; and

$$w_{G'_1}(e) = \begin{cases} w_{G_1}(e), & \text{if } e \in E(G_1); \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to know that G'_1 is 2-connected. Besides, we can see that

$$\max\{d^w(v_1), d^w(v_2), d^w(v_3)\} \geq d$$

for every three pairwise nonadjacent vertices v_1 , v_2 and v_3 in $V(G'_1) \setminus \{x, y\}$. By the induction hypothesis, G'_1 satisfies the conclusion of the theorem.

Suppose that G'_1 contains an (x, y) -path P'_1 of weight at least d . If P'_1 contains neither the added edge xz nor the subpath $xx'z$, then it is an (x, y) -path in G with weight at least d . Otherwise, we can use any (x, z) -path in G_2 to replace the added edge xz or the subpath $xx'z$, and then obtain an (x, y) -path in G with weight at least $w_{G'_1}(P'_1) \geq d$.

Suppose that G'_1 contains an x -path P'_{11} and a y -path P'_{12} which are disjoint and have weight sum at least d . If both these two paths contain none of the added edges in $\{xz, xx', x'z\}$, then they are an x -path and a y -path which are disjoint and have weight sum at least d . Otherwise, if P'_{11} contains either the added edge xz or one of the subpath in $\{xx'z, xzx'\}$, then we can use any (x, z) -path in G_2 to replace it, and obtain an x -path and a y -path which are disjoint and have weight sum at least d . Otherwise, P'_{11} or P'_{12} contains the added edge xx' or zx' . Then we can use either an x -path in G_2 not passing through z or a z -path in G_2 not passing through x to replace it, and obtain an x -path and a y -path which are disjoint and have weight sum at least d .

Suppose that G'_1 contains an x -path P'_{11} and a y -path P'_{12} which are disjoint and contain all vertices of G_1 . If P'_{11} contains the subpath $xx'z$ or xzx' , then we can use the path P'_2 to replace the subpath $xx'z$ or xzx' , and obtain an x -path and a y -path which are disjoint and contain all vertices of G . Otherwise P'_{11} or P'_{12} contains the added edge xx' or zx' , then we can use either $P'_2 - z$ or $P'_2 - x$ to replace it, and obtain an x -path and a y -path which are disjoint and contain all vertices of G .

Case 2.2.2.2. H_2 has only one vertex.

Let x' be the vertex in H_2 . Thus we have $x' \in N(x) \setminus \{y\}$, $d(x') = 2$ and $d^w(x') < d$. By a similar proof, we can obtain that there exists a vertex y' in $N(y) \setminus \{x\}$ such that $d(y') = 2$ and $d^w(y') < d$.

Suppose that $y' = z$. Then $d(z) = 2$. If H_1 has only one vertex y , then the x -path $P_1 = xx'z$ and the y -path $P_2 = y$ are disjoint and contain all vertices of G . Thus, we can assume that H_1 has at least one vertex other than y . Let $G'_1 = G[H_1 \cup \{x\}]$. If $xy \notin E(G'_1)$, we add the edge xy of weight 0 to G'_1 , and then get that G'_1 is 2-connected and $\max\{d_{G'_1}^w(v_1), d_{G'_1}^w(v_2)\} \geq d$ for every pair of nonadjacent vertices v_1 and v_2 in $V(G'_1) \setminus \{x, y\}$. By Theorem 5, there exists an (x, y) -path P'_1 such that either $w_{G'_1}(P'_1) \geq d$ or P'_1 is a Hamilton path in G'_1 . It is obvious that P'_1 is not the added edge xy itself. If $w_{G'_1}(P'_1) \geq d$, then $P = P'_1$ is an (x, y) -path in G with weight at least d . Otherwise P'_1 is a Hamilton path in G'_1 . Then the x -path $P_1 = xx'z$ and the y -path $P_2 = P'_1 - x$ are disjoint and contain all vertices of G .

So, we suppose that $y' \neq z$. Let y'' is the vertex adjacent to y' other than y . We can see that each vertex in $V(G) \setminus \{x, y\}$ other than x', z, y', y'' has weighted degree at least d .

Suppose that $y'' = z$. Then each vertex in $V(G) \setminus \{x, y\}$ other than x', z, y' has weighted degree at least d . Let G' be the weighted graph obtained by adding the edge $x'y'$ of weight 0 to G . Thus,

$$\max\{d_{G'}^w(v_1), d_{G'}^w(v_2)\} \geq d$$

for every pair of nonadjacent vertices v_1 and v_2 in $V(G') \setminus \{x, y\}$. By Theorem 5, there exists an (x, y) -path P' in G' such that either $w_{G'}(P') \geq d$ or P' is a Hamilton path in

G' . If P' does not contain the added edge $x'y'$, then P' is an (x, y) -path in G with weight at least d , or for every $e \in E(P)$, $P' - e$ is the union of an x -path and a y -path which are disjoint and contain all vertices of G . Otherwise, $P' - x'y'$ is an x -path and a y -path which are disjoint and either have weight sum at least d or contain all vertices of G .

Suppose that $y'' \neq z$. Let G'_1 be the weighted graph obtained by adding two edges zy' and zy'' with weight 0 to G_1 . Thus,

$$\max\{d_{G'_1}^w(v_1), d_{G'_1}^w(v_2)\} \geq d$$

for every pair of nonadjacent vertices v_1 and v_2 in $V(G'_1) \setminus \{x, y\}$. By Theorem 5, there exists an (x, y) -path P'_1 in G'_1 such that either $w_{G'_1}(P'_1) \geq d$ or P'_1 is a Hamilton path in G'_1 .

Suppose that $w_{G'_1}(P'_1) \geq d$. If P'_1 contains none of the added edges in $\{xz, zy', zy''\}$, then the path $P = P'_1$ is an (x, y) -path in G with weight at least d . Otherwise we can delete the added edges in P'_1 and then obtain an x -path and a y -path which are disjoint and have weight sum at least d .

Suppose now that P'_1 is a Hamilton path in G'_1 . If P'_1 contains none of the added edges in $\{xz, zy', zy''\}$, or contains only the added edge xz of them, then the x -path $P_1 = xx'$ and the y -path $P_2 = P'_1 - x$ are disjoint and contain all vertices of G . Otherwise, if P'_1 contains the added edge xz and one of the added edges in $\{zy', zy''\}$, then the x -path $P_1 = xx'z$ and the y -path $P_2 = P'_1 - \{x, z\}$ are disjoint and contain all vertices of G . So we assume that P'_1 does not contain the added edge xz . If P'_1 contains one of the added edges e in $\{zy', zy''\}$, then $P'_1 - e$ is the union of an x -path and a y -path which are disjoint and contain all vertices in G'_1 , and one of these two paths is ending in z . Then we can add the edge zx' to it and obtain an x -path and a y -path which are disjoint and contain all vertices of G . If P'_1 contains both the added edges zy' and zy'' , let $P''_1 = P'_1 \cup \{yy'\} - \{zy', zy''\}$, then the x -path $P_1 = xx'z$ and the y -path $P_2 = P''_1 - x$ are disjoint and contain all vertices of G .

The proof is complete. □

5 A related discussion

If we weaken the condition of Theorem 7 by that of large weighted degree sum of four pairwise nonadjacent vertices, the result is also not true.

False Statement 3. *Let G be a 2-connected weighted graph, x be a vertex of G , and d be real number. If $d^w(v_1) + d^w(v_2) + d^w(v_3) + d^w(v_4) \geq 4d$ for every four nonadjacent vertices v_1, v_2, v_3 and v_4 in $V(G) \setminus \{x\}$, then G contains either an x -path of weight at least d or a Hamilton x -path.*

This can be shown as follows. Let H_1 , H_2 and H_3 be three complete graphs on at least three vertices with two vertices x and y in common. Let $G = H_1 \cup H_2 \cup H_3$ and assign weight 0 to every edge of G . Then G is 2-connected and there exist no four pairwise nonadjacent vertices in G . So G satisfies the condition of False Statement 3 for any $d > 0$, but contains neither an x -path of weight at least d nor a Hamilton x -path.

Motivated by Theorems 7 and 8, we pose the following problems.

Problem 1. Let G be a 2-connected weighted graph and d a real number. If $d^w(v_1) + d^w(v_2) + d^w(v_3) + d^w(v_4) \geq 4d$ for every four pairwise nonadjacent vertices v_1, v_2, v_3 and v_4 in $V(G)$, is it true that G contains a path of weight at least d or a Hamilton path?

Problem 2. Let G be a 2-connected weighted graph and d a real number. If $\max\{d^w(v_1), d^w(v_2), d^w(v_3), d^w(v_4)\} \geq d$ for every four pairwise nonadjacent vertices v_1, v_2, v_3 and v_4 in $V(G)$, is it true that G contains a path of weight at least d or a Hamilton path?

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